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# On Fitting $p$ -groups with all proper subgroups satisfying an outer commutator law

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## ABSTRACT

In this note we consider certain Fitting  $p$ -groups in which every proper subgroup satisfies an outer commutator identity and obtained some conditions for such groups to be imperfect. We also give an application of the main theorem to obtain an idea of the abundance of the groups under consideration.

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## 1. Introduction

In [2, Theorem 1.1] the following remarkable result (see also Corollary 2.4) was proved:

**Theorem 1.1.** *Let  $G$  be a minimal non-hypercentral Fitting  $p$ -group. If every proper subgroup is soluble, then  $G$  is soluble.*

It is natural to ask:

*If we replace the “solubility” or “hypercentrality” conditions in the theorem by some other classes of groups what will be the structure of the group?*

For the “hypercentrality” condition, [1, Theorem 2] supplies a special case which can be replaced and [1, Corollary 3.11] gives a substantial generalization of Theorem 1.1.

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In this note we consider both replacements though “solubility” is actually dominant. We shall give a broad generalization of Theorem 1.1 (see Theorem 1.3) considering Theorem 1.2 (Khukhro–Makarenko Theorem) as a key point and exploiting certain arguments in [2].

Now we recall a notion which is necessary in the sequel for the attempt at generalization:

Let  $F(x_1, x_2, \dots)$  be a free group of countable rank, then an *outer commutator of weight 1* is a generator  $x_i$ . An *outer commutator word of weight  $t > 1$*  is a word of the form

$$\omega(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)],$$

where  $u$  and  $v$  are outer commutator words of weight  $r$  and  $t - r$  respectively.

Let  $\omega$  be an outer commutator word of weight  $t$ . By  $\mathfrak{X}_\omega$  we denote the class of groups  $G$  satisfying  $\omega(g_1, \dots, g_t) = 1$  for all  $g_1, \dots, g_t \in G$ . In this case we write

$$\omega(\underbrace{G, \dots, G}_t) = 1.$$

As mentioned above the following theorem provides a key point for success in settling the difficulty in Lemma 2.1.

**Theorem 1.2.** (See [3, Theorem 1], [4, Theorem 1] or [5].) *If a group  $G$  has a subgroup  $H$  of finite index  $n$  satisfying the identity*

$$\omega(\underbrace{H, \dots, H}_w) = 1,$$

where  $\omega$  is an outer commutator word of weight  $w$ , then  $G$  has also a characteristic subgroup  $C$  of finite  $(n, w)$ -bounded index satisfying the same identity

$$\omega(\underbrace{C, \dots, C}_w) = 1.$$

Since every soluble group satisfies an outer commutator identity (see the paragraph after Corollary 2.3), it seems reasonable to consider the classes  $\mathfrak{X}_\omega$ , where  $\omega$  is an outer commutator word of weight  $\geq 2$  in place of the class of all soluble groups of some derived length.

We say that a group  $G$  has the property  $\mathcal{EI}$  if for every finitely generated proper subgroup  $W$  of  $G$  and for every element  $a$  in  $G \setminus W$ , there is a finitely generated subgroup  $V$  containing  $W$ , a generating subset  $Y$  and a proper subgroup  $L$  of  $G$  such that

$$a \in \left( \bigcap_{y \in Y \setminus L} \langle V, y \rangle \right) \setminus V.$$

We call  $Y$  an associated set for  $\mathcal{EI}$  with respect to  $W$  (see [1, Section 3.2]). Clearly in this case  $a \notin V$ , but  $a \in \langle V, y \rangle$  for all  $y \in Y \setminus L$ .

Here is the main result of the present paper.

**Theorem 1.3.** *Let  $G$  be a countable Fitting  $p$ -group with the property  $\mathcal{EI}$  such that  $G$  is the associated set with respect to every finite subgroup. If for every proper subgroup  $K$  of  $G$ , there exists an outer commutator word  $\omega$  of weight  $\geq 2$  such that  $K \in \mathfrak{X}_\omega$ , then  $G' \neq G$ .*

## 2. Proofs of the results

In this section we provide some results mostly relying on the arguments in [2]. In the following lemma we see the importance of having a characteristic subgroup of finite index.

**Lemma 2.1.** *Let  $G$  be a locally nilpotent perfect  $p$ -group. Assume that*

$$G = \bigcup_{\beta < \alpha} N_\beta,$$

*where for each ordinal  $\beta < \alpha$ ,  $N_\beta$  is a proper soluble normal subgroup of  $G$  such that  $N_\beta \leq N_{\beta+1}$ . Let  $U$  be a proper subgroup of  $G$  and assume that for every proper subgroup  $K$  of  $G$ , there exists an outer commutator word  $\omega$  of weight  $\geq 2$  such that  $K \in \mathfrak{X}_\omega$ . Then there exists an ordinal  $\gamma < \alpha$  such that*

$$N_\gamma / (N_\gamma \cap U) N'_\gamma$$

*is infinite.*

**Proof.** Assume that the assertion is false. Then  $|N_\beta : N_\beta \cap U|$  is finite for all  $\beta < \alpha$  by [2, Lemma 2.11]. Put

$$V_\beta := \text{core}_{N_\beta}(N_\beta \cap U).$$

By hypothesis there is an outer commutator word  $\omega$  of weight  $t \geq 2$  such that

$$\omega(\underbrace{U, \dots, U}_t) = 1,$$

i.e.  $U \in \mathfrak{X}_\omega$ . Then clearly  $U \cap N_\beta \in \mathfrak{X}_\omega$  for all  $\beta < \alpha$ . By Theorem 1.2,  $N_\beta$  contains a characteristic subgroup  $W_\beta$  of finite index such that  $W_\beta \in \mathfrak{X}_\omega$ . Put  $\bar{G} = G/W_\beta$ , then  $\bar{G}/C_{\bar{G}}(\bar{N}_\beta)$  is finite. But since  $G$  has no proper subgroup of finite index, we have

$$\bar{G} = C_{\bar{G}}(\bar{N}_\beta),$$

i.e.  $[G, N_\beta] \leq W_\beta$ . It follows that  $N'_\beta \in \mathfrak{X}_\omega$ . Since

$$G = G' = \bigcup_{i=1}^{\infty} N'_\beta,$$

we see  $G \in \mathfrak{X}_\omega$ . But we also have that  $G$  is perfect and thus  $G = 1$ , a contradiction.  $\square$

The following theorem gives a relation between being imperfect and having a subgroup whose index in every proper normal subgroup is finite in some sense.

**Theorem 2.2.** *Let  $G$  be a non-trivial locally nilpotent  $p$ -group with all proper normal subgroups soluble. Assume that for every proper subgroup  $K$  of  $G$  there exists an outer commutator word  $\omega$  of weight  $\geq 2$  such that  $K \in \mathfrak{X}_\omega$ . If  $G$  contains a proper subgroup  $U$  such that  $|N : N \cap U|$  is finite for every proper normal subgroup of  $G$ , then*

$$G \neq G'.$$

Furthermore if  $U \in \mathfrak{X}_u$  for some outer commutator word  $u$  of weight  $\geq 2$ , then  $\gamma_3(G) \in \mathfrak{X}_u$ .

**Proof.** Assume for a contradiction that  $G$  is perfect. Since  $G$  is locally nilpotent and has no proper subgroup of finite index,  $G$  is a union of a chain of proper normal subgroups of  $G$ . But by Lemma 2.1,  $G$  has a proper normal subgroup  $M$  such that  $M/(M \cap U)M'$  and hence  $|M : M \cap U|$  is infinite, and this is a contradiction. Consequently, we have  $G \neq G'$ . Now since  $U \in \mathfrak{X}_u$ , so is  $G' \cap U$ . Also  $|G' : G' \cap U|$  is finite, and so by Theorem 1.2  $G'$  has a characteristic subgroup  $V \in \mathfrak{X}_u$  such that  $G'/V$  is finite. Put  $\bar{G} := G/V$ , then  $\bar{G}/C_{\bar{G}}(\bar{G}')$  is finite. But since  $\bar{G}$  has no proper subgroup of finite index,

$$\bar{G} = C_{\bar{G}}(\bar{G}').$$

So  $[G, G'] \leq V$ , i.e.  $\gamma_3(G) \in \mathfrak{X}_u$ .  $\square$

Now we embark on the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Assume for a contradiction that  $G$  is perfect. Since  $G$  is countable, we can find an ascending series

$$F_1 \leq F_2 \leq \dots \leq F_i \leq \dots$$

of finite subgroups of  $G$  whose union is  $G$ . Now put  $N_i := F_i^G$  for every  $i \geq 1$ . Since  $G$  is a Fitting group, each  $N_i$  is nilpotent of finite exponent [7, 5.2.5]. We also have that

$$G = \bigcup_{i=1}^{\infty} N_i.$$

Now let  $a$  be a non-trivial element of  $G$ . Then since  $G$  has  $\mathcal{EI}$ ,  $G$  contains a finite subgroup  $V$  and a proper subgroup  $L$  such that  $a \notin V$  but  $a \in \langle V, y \rangle$  for all  $y \in G \setminus L$ . Put  $W = \langle V, a \rangle^G L$ .

By Lemma 2.1, there exists a positive integer  $s$  such that  $N_s/(N_s \cap W)N'_s$  is infinite. Put  $S = N_s \langle V, a \rangle$ , then  $S$  is nilpotent and  $S/(S \cap W)S'$  is infinite by [2, Lemma 2.13], since  $N_s$  is a normal subgroup and  $\langle V, a \rangle$  is a finite subgroup of  $G$  such that  $\langle V, a \rangle^G \leq W$ . Now put

$$R/(S \cap W)S' := \text{Frat}(S/(S \cap W)S').$$

Since  $S$  has finite exponent, we have  $S/R$  is infinite elementary abelian. By [6, Satz (6)],  $S$  has a subgroup  $T$  such that  $a \notin T$ ,  $V \leq T$  and  $TR/R$  is infinite. Choose  $y \in T \setminus R$ . Since  $T \cap W \leq R$ , we see that  $y \notin L$  and  $a \in \langle V, y \rangle$ , a contradiction.  $\square$

The following is a useful corollary to Theorem 1.3.

**Corollary 2.3.** Let  $G$  be a minimal non-hypercentral Fitting  $p$ -group. If for every proper subgroup  $K$  of  $G$  there exists an outer commutator word  $\omega$  of weight  $\geq 2$  such that  $K \in \mathfrak{X}_{\omega}$ , then  $G' \neq G$ .

**Proof.** Assume that  $G$  is perfect. So  $Z(G/Z(G)) = 1$  and we may assume that  $G$  has trivial center and has no proper subgroup of finite index. By [2, Lemma 2.3],  $G$  has  $\mathcal{EI}$ . Since the class of hypercentral groups is countably recognizable,  $G$  is countable. So the hypotheses of Theorem 1.3 hold and consequently we have that  $G$  is not perfect. This contradiction completes the proof.  $\square$

Define the outer commutator words  $\phi_i$  ( $i \geq 0$ ) as follows:  $\phi_0(x) = x$  and for  $i \geq 1$ ,

$$\phi_i(x_1, \dots, x_{2^i}) = [\phi_{i-1}(x_1, \dots, x_{2^{i-1}}), \phi_{i-1}(x_{2^{i-1}+1}, \dots, x_{2^i})].$$

Then clearly  $G$  is soluble of derived length  $\leq n$  if and only if  $\phi_n(x_1, \dots, x_{2^n}) = 1$ .

Now Theorem 1.1 is seen to be a special case of Corollary 2.3:

**Corollary 2.4.** *Let  $G$  be a minimal non-hypercentral Fitting  $p$ -group. If every proper subgroup of  $G$  is soluble, then  $G$  is soluble.*

**Proof.** If  $H$  is a proper subgroup of derived length  $m$ , then  $H \in \mathfrak{F}_{\phi_m}$ . So by Corollary 2.3, we have  $G' \neq G$ . Consequently  $G$  is soluble, since  $G'$  is soluble.  $\square$

### 3. An application of Theorem 1.3

In this section we will consider a class of groups in place of the class of all hypercentral groups. Though we provide only one application, readers may have an idea to find some other useful classes of groups.

Define the outer commutator words  $\theta_i$  for  $i \geq 0$  as follows:  $\theta_0(x) = x$ ,  $\theta_1(x_1, x_2) = [x_1, x_2]$  and for  $k \geq 2$

$$\theta_k(x_1, \dots, x_k, x_{k+1}) = \begin{cases} [x_{k+1}, \theta_{k-1}(x_1, \dots, x_k)], & \text{if } k \text{ is even;} \\ [\theta_{k-1}(x_1, \dots, x_k), x_{k+1}], & \text{if } k \text{ is odd.} \end{cases}$$

As is seen above, for  $k$  even and  $k$  odd one commutator is just the inverse of the other.

Let  $\mathfrak{K}$  be the class of groups  $G$  which satisfy the following condition:

For every sequence  $x_1, x_2, \dots, x_i, \dots$ , there is a positive integer  $n$  such that

$$\theta_n(x_1, \dots, x_{n+1}) = 1.$$

**Lemma 3.1.** *Let  $G$  be a perfect locally nilpotent  $p$ -group with every proper subgroup a  $\mathfrak{K}$ -group. Then  $G$  satisfies the property  $\mathcal{ET}$ .*

**Proof.** Assume that the assertion is false and let  $Y$  be a generating subset of  $G$ . Then there exists a finite subgroup  $W$  of  $G$ , an element  $a \in G \setminus W$  such that  $a \notin \langle W, y \rangle$  for every  $y \in Y \setminus Z(G)$ . So we have an element  $y_1 \in Y \setminus L$  such that  $a \notin \langle W, y_1 \rangle$ . Now clearly  $\theta_0(y_1) = y_1 \notin Z(G)$ .

Since  $G$  is perfect, we have  $Z(G/Z(G)) = 1$ . Put  $W_1 := \langle W, y_1 \rangle$ . By assumption  $a \notin \langle W_1, y \rangle$  for every  $y \in Y \setminus C_G(y_1 Z(G))$ . So there exists an element  $y_2 \in Y \setminus C_G(y_1 Z(G))$  such that  $a \notin \langle W_1, y_2 \rangle$ . Put  $W_2 := \langle W_1, y_2 \rangle$ . We then have  $\theta_1(y_1, y_2) \notin Z(G)$ . Now assume we have found elements  $y_1, \dots, y_n$  such that  $a \notin \langle W, y_1, \dots, y_n \rangle$  and  $\theta_{n-1}(y_1, \dots, y_n) \notin Z(G)$ . Put

$$W_n := \langle W, y_1, y_2, \dots, y_n \rangle.$$

So by assumption we have  $a \notin \langle W_n, y \rangle$  for every  $y \in Y \setminus C_G(\theta_{n-1}(y_1, \dots, y_n) Z(G))$ . So again there is an element

$$y_{n+1} \in Y \setminus C_G(\theta_{n-1}(y_1, \dots, y_n) Z(G))$$

such that  $a \notin \langle W_n, y_{n+1} \rangle$ . Then if  $n$  is even we have

$$\theta_n(y_1, \dots, y_{n+1}) = [y_{n+1}, \theta_{n-1}(y_1, \dots, y_n)] \notin Z(G)$$

and if  $n$  is odd we have

$$\theta_n(y_1, \dots, y_{n+1}) = [\theta_{n-1}(y_1, \dots, y_n), y_{n+1}] \notin Z(G).$$

Put  $W_{n+1} := \langle W_n, y_{n+1} \rangle$ . Now consider

$$X := \bigcup_{i=1}^{\infty} W_i,$$

then since  $a \notin X$ ,  $X \neq G$ . Now for every  $i \geq 1$ ,  $\theta_i(y_1, \dots, y_{i+1}) \neq 1$  and thus we reach a contradiction. So the result follows.  $\square$

**Corollary 3.2.** *Let  $G$  be a minimal non- $\mathfrak{R}$  Fitting  $p$ -group. If for every proper subgroup  $K$  of  $G$  there exists an outer commutator word  $\omega$  of weight  $\geq 2$  such that  $K \in \mathfrak{X}_\omega$ , then  $G' \neq G$ .*

**Proof.** Assume that  $G$  is perfect. So  $Z(G/Z(G)) = 1$  and we may assume that  $G$  has trivial center and has no proper subgroup of finite index. By Lemma 3.1,  $G$  has  $\mathcal{ET}$ .

Now assume that  $G$  is uncountable. For every  $n \geq 2$  we can find elements

$$y_{1,n}, \dots, y_{n+1,n} \in G \quad \text{such that} \quad \theta_n(y_{1,n}, \dots, y_{n+1,n}) \neq 1.$$

Then the subgroup

$$Y := \langle y_{1,i}, \dots, y_{i+1,i} : i \geq 1 \rangle$$

is countable, but  $Y \notin \mathfrak{R}$ . This contradiction shows that  $G$  is countable. So the hypothesis of Theorem 1.3 holds and it follows that  $G$  is not perfect, a contradiction. So the proof is complete.  $\square$

As such application, it can be seen that there are numerous applications of Theorem 1.3. Following similar proofs to that of Lemma 3.1, one can see the impossibility of being perfect of certain structures (see for example  $\mathfrak{R}$ -groups defined in [1] as another application).

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